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On using materially conserved quantities to construct solutions of differential equations: II. Non-ideal systems

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Abstract. In certain fluid-dynamical systems sufficient materially conserved quantities (MCQs) can be found for the construction of a first integral of the system. The first integral is formed by functionally relating the MCQs. Such integrals simplify construction of solutions of the systems greatly, though determination of the function relating the MCQs can be problematic. It is often prescribed. Here we consider generalizations of such systems, for example diffusion is added. Results indicate that whilst MCQs exist for some such perturbations these are rare. We therefore suggest an alternative approach for constructing solution ansätze: an ansatz is made based upon that for the unperturbed system but in which the function relating the MCQs changes 'slowly' with each independent variable. Particular examples are considered for the two-dimensional Euler equations and for the equations describing large-scale, steady flow of a thin layer of fluid on the surface of a rotating sphere at mid latitudes (applicable to ocean dynamics). In each case, as a result of the ansatz we find equations which, given appropriate boundary conditions, determine the relating-function.

1. Introduction

A materially conserved quantity (MCQ) of a fluid-dynamical system is defined as any quantity q satisfying

$$\frac{\mathrm{D}q}{\mathrm{D}t} \equiv \frac{\partial q}{\partial t} + u^i \frac{\partial q}{\partial x^i} = 0 \tag{1.1}$$

for example in the case of Cartesian coordinates in three dimensions,

$$\frac{\partial q}{\partial t} + u\frac{\partial q}{\partial x} + v\frac{\partial q}{\partial y} + w\frac{\partial q}{\partial z} = 0 \tag{1.2}$$

where $\boldsymbol{u} = (u, v, w)$ is the velocity vield, for *any* flow satisfying the corresponding governing equations. Recently Hood (1998) showed how one may systematically determine such MCQs of some fluid-dynamical systems. These MCQs can be used to construct quite general, exact, analytical solutions of the corresponding equations: given n MCQs of a system of PDEs in n independent variables the MCQs are necessarily functionally related. These relations—first integrals of the system—take the form of differential equations which are (usually) significantly simpler than the given (governing) equations and so may be considered as solution ansätze for the system. Here we study what happens when the equations are perturbed by the introduction of, say, diffusion. Can modified MCQs be constructed? Or can one modify the solution ansätze in some way so that the original MCQs may still be used, albeit in an approximate rather than exact method? It turns out that the latter approach is the more fruitful.

We focus our attention on two fluid systems. First, the two-dimensional (2D) Euler equations,

$$\begin{aligned} uu_x + vu_y &= -p_x \\ uv_x + vv_y &= -p_y \end{aligned} \tag{1.3}$$

in which $\boldsymbol{u} = (u, v)$ is the velocity field and p is pressure, which describe steady fluid flow in the limit in which viscosity is zero. It is well known (e.g., Batchelor 1967) that (1.3) satisfy two material conservation laws: both the stream function, ψ, defined by

$$(u, v) = (-\psi_y, \psi_x) \tag{1.4}$$

and vorticity, ω, defined by

$$\omega = -\psi_{xx} - \psi_{yy} \tag{1.5}$$

are materially conserved, and since we have two MCQs and a 2D system then a first integral of (1.3) is

$$\omega = \mathcal{F}(\psi) \tag{1.6}$$

where $\mathcal{F}$ is arbitrary. Any solution of (1.6), for any $\mathcal{F}$, is a solution of the 2D-Euler equations, (1.3). Given $\mathcal{F}$ one can solve (1.6) for ψ and hence obtain the velocity and pressure fields. But what is $\mathcal{F}$? (It cannot always be determined by far-field, e.g., upstream, boundary conditions.)

Secondly, we consider the equations describing large-scale, steady flow of a thin layer of fluid on the surface of a rotating sphere at mid latitudes. (See, for example, the classic papers by Robinson and Stommel (1959) and Welander (1959, 1971); see also Needler (1971), and more recently Hood and Williams (1996) and references therein.) The primitive equations describing ocean dynamics are the compressible Navier–Stokes equations in a rotating reference frame, a conservation of mass equation and an equation of state. At mid latitudes scaling analysis suggests (e.g., Pedlosky 1986) that the simplest model one might use is

$$u = -\frac{p_y}{y} \qquad v = \frac{p_x}{y} \qquad \rho = p_z \qquad u_x + v_y + w_z = 0 \tag{1.7a}$$

with

$$u\rho_x + v\rho_y + w\rho_z = 0 \tag{1.7b}$$

where u, v and w are the components of the velocity field in the x, y and z directions respectively, ρ is density, and p is pressure. (The first two equations show a counter-intuitive balance between gradients of the pressure field and the velocity field; the third equation represents hydrostatic balance, the fourth is continuity and the final equation represents advection of the density field by the velocity field.) These are often called the *thermocline equations*.

It is useful to rewrite the thermocline equations in terms of a potential, $M(x, y, z)$ (Welander, 1959), namely,

$$M_x M_{zzz} + y(M_{xz}M_{yzz} - M_{yz}M_{xzz}) = 0 \tag{1.8a}$$

$$u = -\frac{M_{yz}}{y} \qquad v = \frac{M_{xz}}{y} \qquad w = \frac{M_x}{y^2} \qquad p = M_z \qquad \rho = M_{zz}. \tag{1.8b}$$

This format was used by Hood (1998) in which a systematic method was described for computing materially conserved quantities of continuum dynamical systems. (1.7) admits three materially conserved quantities and in terms of M these are

$$\rho = M_{zz} \qquad B = M_z - zM_{zz} \qquad \text{and} \qquad q = yM_{zzz} \tag{1.9}$$

density, the Bernoulli functional and potential vorticity, respectively. (Conservation of B can be thought of as conservation of energy by each fluid parcel; the role and significance of potential vorticity is described by many texts on geophysical fluid dynamics, e.g., Pedlosky (1986).)

In general density is a monotonically increasing function of depth within the ocean (this not true everywhere). Assuming this, one can choose ρ as the 'vertical' independent variable and choosing B as the dependent variable one can then write the equations in a particularly simple form:

$$yu = -B_y \qquad yv = B_x \qquad z = B_\rho \qquad w = uz_x + vz_y$$
$$(uz_\rho)_x + (vz_\rho)_y = 0 \tag{1.10a}$$

together with

$$y(B_x B_{y\rho\rho} - B_y B_{x\rho\rho}) - B_x B_{\rho\rho} = 0. \tag{1.10b}$$

(Details are given by Janowitz (1986).) Since this representation of the dynamics is simpler we use it below. (Unfortunately the boundary conditions are now significantly more difficult to handle e.g., Killworth (1987).) In these *isopycnal coordinates* the three MCQs are

$$\rho \quad B \quad q = \frac{y}{B_{\rho\rho}} \tag{1.11}$$

and since we have three MCQs and a 3D system then a first integral of the system (1.10a) and (1.10b) is

$$q = \mathcal{Q}(B, \rho). \tag{1.12}$$

Given $\mathcal{Q}$ one has to solve only an *ordinary* differential equation for B, from which the velocity and other fields can be obtained. But what is $\mathcal{Q}$?

In this paper we give an answer to the questions posed above—what are $\mathcal{F}$ and $\mathcal{Q}$? First, in section 2 we look for exact MCQs of perturbations of both the 2D-Euler system and the ideal thermocline system: results indicate that whilst MCQs exist for some perturbations such existence is likely to be rare. Motivated by this we adopt a more pragmatic approach in section 3: we suggest that whilst the addition of diffusion, for example, to a system such as the 2D-Euler or ideal thermocline equations is singular in our usual coordinates it is regular in MCQ-space. This leads to the solution ansätze

$$\omega = \mathcal{F}(\psi, X, Y) \tag{1.13}$$

(cf (1.6)) for the 2D-Euler equations and

$$q = \mathcal{Q}(B, \rho, X, Y) \tag{1.14}$$

(cf (1.12)) for the thermocline equations, where

$$X = \delta_1 x \qquad Y = \delta_2 y \qquad \mathrm{O}(\delta_1) = \mathrm{O}(\delta_2) = \mathrm{O}(\kappa). \tag{1.15}$$

Using these ansätze analytical progress is made in three examples. In the first the problem of nearly inviscid, steady 2D flow within closed streamlines discussed by Batchelor (1956) is revisited. Batchelor's result is recovered and, in addition, a correction term suggested. In the second and third models of the dynamics underlying the (unexpected) very nonlinear temperature profile in the worlds oceans are investigated. Generalizations of these models are obtained. In each of these three examples, by means of the solution ansätze given above, we obtain equations sufficient to determine the unknown function relating the MCQs. In the first case the function is found explicitly. Finally, the wider significance of the results and possible generalizations are discussed in section 4.

Notation. Subscripts are used to denote partial derivatives, for example, A_x denotes the x partial derivative of A; $A_{[x]}$ denotes the total derivative, for example, if A depends on the independent variables, on a function ψ and on the ψ-derivatives then

$$A_{[x]} \equiv A_x + A_\psi \psi_x + A_{\psi_x} \psi_{xx} + \cdots. \tag{1.16}$$

2. Materially conserved quantities of perturbed systems

In this section we attempt to find quantities which are exactly materially conserved by flows described by perturbations of the 2D-Euler and ideal thermocline equations. The method used is systematic and fully described by Hood (1998). Briefly: an ansatz is made, for example in the context of the 2D-Euler equations,

$$q = \mathcal{Q}(x, y, \psi, \psi_x, \psi_y, \psi_{xx}, \psi_{xy}, \psi_{yy}) \tag{2.1}$$

so that q is a second-order MCQ—depends at most on second-order derivatives of ψ; this is substituted into (1.1) and noting that the result must be satisfied *for all* functions ψ—which means that distinct derivatives of ψ are independent—leads to constraints on $\mathcal{Q}$ which are solved where possible. Throughout the computation one must take into account the *frame* of the governing equations, that is the equations themselves and appropriate differential consequences.

2.1. The 2D-Euler equations

In this section we seek materially conserved quantities of perturbations of the steady, 2D-Euler equations, (1.3). These equations describe steady fluid flow at large Reynolds number, i.e., when the effects of viscosity may be neglected. In particular we consider the case in which (small) viscosity effects are reintroduced into the system.

We consider perturbations of the form

$$uu_x + vu_y = R(x, y, \psi, \psi_x, \psi_y, \psi_{xx}, \ldots) \tag{2.2a}$$
$$uv_x + vv_y = S(x, y, \psi, \psi_x, \psi_y, \psi_{xx}, \ldots) \tag{2.2b}$$

i.e.,

$$\psi_y\psi_{xy} - \psi_x\psi_{yy} = R(x, y, \psi, \psi_x, \psi_y, \psi_{xx}, \ldots) \tag{2.3a}$$
$$-\psi_y\psi_{xx} + \psi_x\psi_{xy} = S(x, y, \psi, \psi_x, \psi_y, \psi_{xx}, \ldots) \tag{2.3b}$$

in which neither R nor S are necessarily small and look for quantities $Q(x, y, \psi, \psi_x, \ldots)$ which satisfy (1.1) which in this context becomes

$$-\psi_y \mathcal{Q}_{[x]} + \psi_y \mathcal{Q}_{[y]} = 0 \tag{2.4}$$

or, expanding the total derivatives

$$\begin{aligned}\psi_x\{\mathcal{Q}_x &+ \mathcal{Q}_\psi \psi_x + \mathcal{Q}_{\psi_x}\psi_{xx} + \mathcal{Q}_{\psi_y}\psi_{xy} + \mathcal{Q}_{\psi_{xx}}\psi_{xxx} + \mathcal{Q}_{\psi_{xy}}\psi_{xxy} \\ &+\mathcal{Q}_{\psi_{yy}}\psi_{xyy} + \cdots\} + \psi_y\left\{Q_y + Q_\psi\psi_y + Q_{\psi_x}\psi_{xy} + Q_{\psi_y}\psi_{yy} + Q_{\psi_{xx}}\psi_{xxy}\right. \\ &\left.+Q_{\psi_{xy}}\psi_{xyy} + Q_{\psi_{yy}}\psi_{yyy} + \cdots\right\} = 0.\end{aligned} \tag{2.5}$$

Note, ψ is materially conserved for *any* perturbation of this form.

Analysis of the general dependence of R and S on third- (or higher-) order derivatives of ψ with general q is not simple! Instead we consider two examples: we focus on MCQs which depend on derivatives of ψ no higher than second order. Given these restrictions we need only consider the part of the frame of (2.3) consisting of the equations themselves and the first-order derivatives, i.e.,

$$\psi_{xy}^2 + \psi_y\psi_{xxy} - \psi_{xx}\psi_{yy} - \psi_x\psi_{xyy} = R_{[x]} \tag{2.6a}$$

$$\psi_y\psi_{xyy} - \psi_x\psi_{yyy} = R_{[y]} \tag{2.6b}$$

$$\psi_x\psi_{xxy} - \psi_y\psi_{xxx} = S_{[x]} \tag{2.6c}$$

$$\psi_{xy}^2 + \psi_x\psi_{xyy} - \psi_{yy}\psi_{xx} - \psi_y\psi_{xxy} = S_{[y]} \tag{2.6d}$$

the x and y derivatives of (2.3*a*), and the x and y derivatives of (2.3*b*), respectively. In general we would expect to be able to solve (2.6) for the third-order ψ derivatives in terms of lower order terms—the uniqueness and existence of such a solution plays a significant rôle in the search for MCQs—and hence eliminate each from (2.5).

Example 2.1. *We suppose that both R and S are independent of second- and higher-order derivatives of ψ, i.e., can depend on the velocity field but not its derivatives:*

$$uu_x + vu_y = R(x, y, \psi, \psi_x, \psi_y) \tag{2.7a}$$

$$uv_x + vv_y = S(x, y, \psi, \psi_x, \psi_y). \tag{2.7b}$$

In this case (2.6) are not consistent unless

$$R_{[x]} + S_{[y]} + 2\psi_{xx}\psi_{yy} - 2\psi_{xy}^2 = 0. \tag{2.8}$$

Given (2.8) we can solve (2.6) for (just) three of the third-order derivatives of ψ in terms of the fourth. We then expect to eliminate these three from (2.5) which would then partition into two constraints: the coefficient of the fourth third-order derivative of ψ and the remaining terms. Eliminating ψ_{xxx}, ψ_{xxy} and ψ_{yyy} we find

$$\begin{aligned}
&\psi_x\{Q_y + Q_{\psi_x}\psi_{xy} + Q_{\psi_y}\psi_{yy}\}\\
&-\psi_y\{Q_x + Q_{\psi_x}\psi_{xx} + Q_{\psi_y}\psi_{xy}\}\\
&+Q_{\psi_{xx}}S_{[x]} - (R_{[x]} + \psi_{xx}\psi_{yy} - \psi_{xy}^2)Q_{\psi_{xy}} - Q_{\psi_{yy}}R_{[y]} = 0.
\end{aligned} \tag{2.9}$$

Fortuitously the terms involving ψ_{xyy} have cancelled. It remains to solve (2.9) simultaneously with (2.7a) and (2.7b). Given R and S any solution of (2.9) yields a second MCQ.

Particular examples are readily constructed. For example:

(a) If $R = -p_x + \lambda v + f(x)$ and $S = -p_y - \lambda u + g(y)$, where λ is an arbitrary constant, and both f and g are arbitrary functions, then ω is still a MCQ.

(b) If $S = \mathrm{e}^{-x}$ (so that the p_y is negligible) then $Q = \psi_x + \psi_{xx}$ is a second MCQ—R remains any function of x, y, ψ, ψ_x and ψ_y (in particular, p_x is not required to be negligible).

Example 2.2. *We consider the usual parametrization of viscosity:*

$$uu_x + vu_y = -p_x + \nu(u_{xx} + u_{yy}) \tag{2.10a}$$

$$uv_x + vv_y = -p_y + \nu(v_{xx} + v_{yy}) \tag{2.10b}$$

ν a constant, i.e.,

$$\psi_y\psi_{xy} - \psi_x\psi_{yy} = -p_x - \nu(\psi_{xxy} + \psi_{yyy}) \tag{2.11a}$$

$$-\psi_y\psi_{xx} + \psi_x\psi_{xy} = -p_y + \nu(\psi_{xxx} + \psi_{xyy}). \tag{2.11b}$$

Again looking for MCQs which depend upon at most second-order derivatives of ψ then the appropriate part of the frame of (2.10) is simply the equations themselves. We choose to eliminate ψ_{xxx} and ψ_{yyy} from (2.5) by use of (2.11); ψ_{xxy} and ψ_{xyy} remain, and since Q is independent of each their coefficients must each equal zero. We obtain

$$-\psi_y Q_{\psi_{xy}} + \psi_x(Q_{\psi_{xx}} - Q_{\psi_{yy}}) = 0 \tag{2.12a}$$

$$\psi_y(Q_{\psi_{xx}} - Q_{\psi_{yy}}) + \psi_x Q_{\psi_{xy}} = 0 \tag{2.12b}$$

respectively. From these we find

$$\mathcal{Q}_{\psi_{xy}} = 0 \tag{2.13}$$

and

$$\mathcal{Q}_{\psi_{xx}} = \mathcal{Q}_{\psi_{yy}}. \tag{2.14}$$

Since we now know that $\mathcal{Q}$ is independent of ψ_{xy} the remaining terms split into two parts, the coefficient of ψ_{xy} and the rest, respectively,

$$-\psi_y(\nu\mathcal{Q}_{\psi_y} + \psi_x\mathcal{Q}_{\psi_{xx}}) + \psi_x\left(\nu\mathcal{Q}_{\psi_x} - \psi_y\mathcal{Q}_{\psi_{yy}}\right) = 0 \tag{2.15a}$$

$$-\psi_y\{(\mathcal{Q}_x + \mathcal{Q}_{\psi_x}\psi_{xx})\nu + \mathcal{Q}_{\psi_{xx}}(p_y - \psi_y\psi_{xx})\}$$
$$+\psi_x\{(\mathcal{Q}_y + \mathcal{Q}_{\psi_y}\psi_{yy})\nu - \mathcal{Q}_{\psi_{yy}}(p_x + \psi_x\psi_{yy})\} = 0. \tag{2.15b}$$

Consider (2.15b): the only way to eliminate the derivatives of p is to set $\mathcal{Q}_{\psi_{xx}} = \mathcal{Q}_{\psi_{yy}} = 0$ from which it immediately follows that $\mathcal{Q}_{\psi_x} = \mathcal{Q}_{\psi_y} = 0$, and in turn that $\mathcal{Q}_x = \mathcal{Q}_y = 0$. There is no second MCQ in this case.

We should first take note that example 2.1 tells us that second MCQs do exist for R and S different from $-p_x$ and $-p_y$, respectively, in particular ω is still an MCQ in some cases—this is by no means obvious. (Recall that ψ is always materially conserved.) One might argue that the results from this example are somewhat artificial. In example 2.2 we considered a perturbation based on the addition of Fickian diffusion to the 2D-Euler equations and found that no second MCQ of second order (or lower) exists. Of course a second MCQ of higher-order might exist, however, our aim is to find MCQs which are of *practical* use in constructing analytical solutions of the (perturbed) 2D-Euler equations: it seems doubtful that solution ansätze involving higher order MCQs would *simplify* their solution.

2.2. *The thermocline equations*

We next use the isopycnal formulation of the thermocline equations, (1.10), to seek MCQs of a modified thermocline system in which

$$y(B_xB_{y\rho\rho} - B_yB_{x\rho\rho}) - B_xB_{\rho\rho} = \kappa y^2 B_{\rho\rho\rho} \tag{2.16}$$

where κ is assumed constant. We have introduced a diffussive term into the equations.

Before doing so it is worth digressing briefly: what does this 'vertical' diffusion add to the system and why choose this particular parametrization of diffusion? It is argued by Gent and McWilliams (1990) that if one does not resolve small-scale motion (eddies) in a model of ocean dynamics then, even if the model is adiabatic, density should not be preserved, i.e.,

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = K \tag{2.17}$$

where K is to be determined. Therefore our material derivative becomes

$$\frac{\mathrm{D}}{\mathrm{D}t} = \boldsymbol{u}\cdot\boldsymbol{\nabla}_\rho + K\frac{\partial}{\partial\rho} \qquad \boldsymbol{\nabla}_\rho \equiv \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \tag{2.18}$$

and *requiring* that potential vorticity is still a MCQ, from (1.1), (1.10a) and (1.11),

$$\left(-\frac{B_x}{y}\frac{\partial}{\partial x} + \frac{B_y}{y}\frac{\partial}{\partial y} + K\frac{\partial}{\partial\rho}\right)\frac{y}{B_{\rho\rho}} = 0. \tag{2.19}$$

Expanding (2.19) and setting $K = -\kappa$ yields (2.16).

We now show that there are no materially conserved quantities of third order and below except potential-vorticity, i.e., that for the dynamics described by (2.16) with (1.10*a*), the only differential quantity of the form

$$q = \mathcal{Q}(x, y, \rho, B, B_x, \ldots, B_{y\rho\rho}, B_{\rho\rho\rho}) \tag{2.20}$$

satisfying (1.1), in which the material-derivative is given by (2.18), is $q = Q(y/B_{\rho\rho})$ where Q is arbitrary.

The appropriate part of the frame of (2.16) is the equation itself plus the three first-order partial derivatives:

$$(B_{xx}B_{y\rho\rho} + B_x B_{xy\rho\rho} - B_{xy}B_{x\rho\rho} - B_y B_{xx\rho\rho})y - B_{xx}B_{\rho\rho} - B_x B_{x\rho\rho} = \kappa y^2 B_{x\rho\rho\rho} \tag{2.21a}$$

$$\begin{aligned}(B_{xy}B_{y\rho\rho} &+ B_x B_{yy\rho\rho} - B_{yy}B_{x\rho\rho} - B_y B_{xy\rho\rho})y \\ &+ B_x B_{y\rho\rho} - B_y B_{x\rho\rho} - B_{xy}B_{\rho\rho} - B_x B_{y\rho\rho} = 2\kappa y B_{\rho\rho\rho} + \kappa y^2 B_{y\rho\rho\rho}\end{aligned} \tag{2.21b}$$

$$(B_{x\rho}B_{y\rho\rho} + B_x B_{y\rho\rho\rho} - B_{y\rho}B_{x\rho\rho} - B_y B_{x\rho\rho\rho})y - B_{x\rho}B_{\rho\rho} - B_x B_{\rho\rho\rho} = \kappa y^2 B_{\rho\rho\rho\rho}. \tag{2.21c}$$

There are two significant points to note before beginning the computation. First, we can use (2.16) to eliminate $B_{\rho\rho\rho}$ in terms of other third-order and lower-order derivatives wherever it appears, i.e., we may assume that $\mathcal{Q}_{B_{\rho\rho\rho}} = 0$ without loss of generality. Secondly, we can use (2.21*a*) and (2.21*b*) to eliminate, say, $B_{xx\rho\rho}$ and $B_{yy\rho\rho}$ in terms of other derivatives of B, but we cannot use (2.21*c*): so doing would introduce $B_{\rho\rho\rho\rho}$ which appears nowhere else as we have taken $\mathcal{Q}$ to be independent of $B_{\rho\rho\rho}$.

Substituting (2.20) into (1.1) and using (2.18) we obtain an equation which $\mathcal{Q}$ must satisfy for all functions B (cf 2.5). Since $\mathcal{Q}$ does not depend upon the fourth-order derivatives of B the coefficients of each of these in this equation must be zero. Making use of the frame of (2.16) as just described then the coefficients of B_{xxxx}, B_{xxxy}, B_{xxyy}, B_{xyyy}, B_{yyyy}, $B_{xxx\rho}$, $B_{xxy\rho}$, $B_{xyy\rho}$ and $B_{yyy\rho}$ are, respectively,

$$-\frac{B_y}{y}\mathcal{Q}_{B_{xxx}} = 0 \tag{2.22a}$$

$$-\frac{B_y}{y}\mathcal{Q}_{B_{xxy}} = 0 \tag{2.22b}$$

$$-\frac{B_y}{y}\mathcal{Q}_{B_{xyy}} + \frac{B_x}{y}\mathcal{Q}_{B_{xxy}} = 0 \tag{2.22c}$$

$$-\frac{B_y}{y}\mathcal{Q}_{B_{yyy}} + \frac{B_x}{y}\mathcal{Q}_{B_{xyy}} = 0 \tag{2.22d}$$

$$\frac{B_x}{y}\mathcal{Q}_{B_{yyy}} = 0 \tag{2.22e}$$

$$-\frac{B_y}{y}\mathcal{Q}_{B_{xx\rho}} = 0 \tag{2.22f}$$

$$-\frac{B_y}{y}\mathcal{Q}_{B_{xy\rho}} + \frac{B_x}{y}\mathcal{Q}_{B_{xx\rho}} - \kappa\mathcal{Q}_{B_{xxy}} = 0 \tag{2.22g}$$

$$-\frac{B_y}{y}\mathcal{Q}_{B_{yy\rho}} + \frac{B_x}{y}\mathcal{Q}_{B_{xy\rho}} - \kappa\mathcal{Q}_{B_{xyy}} = 0 \tag{2.22h}$$

$$\frac{B_x}{y}\mathcal{Q}_{B_{yy\rho}} = 0 \tag{2.22i}$$

(recall that $B_{xx\rho\rho}$ and $B_{yy\rho\rho}$ have been eliminated, and $B_{\rho\rho\rho\rho}$ does not appear); the coefficients of $B_{xy\rho\rho}$, $B_{x\rho\rho\rho}$ and $B_{y\rho\rho\rho}$ are identically zero. From (2.22) we conclude that

$$\mathcal{Q}_{B_{xxx}} = \mathcal{Q}_{B_{xxy}} = \mathcal{Q}_{B_{xyy}} = \mathcal{Q}_{B_{yyy}} = \mathcal{Q}_{B_{xx\rho}} = \mathcal{Q}_{B_{xy\rho}} = \mathcal{Q}_{B_{yy\rho}}. \tag{2.23}$$

It remains to determine the dependence of $\mathcal{Q}$ on $B_{x\rho\rho}$ and on $B_{y\rho\rho}$, and of course the lower-order derivatives of B.

Since we now know that $\mathcal{Q}$ depends on at most only two third-order derivatives of B, $B_{x\rho\rho}$ and $B_{y\rho\rho}$, we know that the coefficients of all other third-order derivatives of B must be zero. Setting the coefficients of B_{xxx}, B_{xxy}, B_{xyy}, B_{yyy}, $B_{xx\rho}$, $B_{xy\rho}$ and $B_{yy\rho}$ each to zero we obtain a system similar to (2.22) from which it is easy to show that

$$\mathcal{Q}_{B_{xx}} = \mathcal{Q}_{B_{xy}} = \mathcal{Q}_{B_{yy}} = \mathcal{Q}_{B_{x\rho}} = \mathcal{Q}_{B_{y\rho}} = 0. \tag{2.24}$$

It remains to determine the dependence of $\mathcal{Q}$ on $B_{\rho\rho}$.

Collecting results we have

$$\begin{aligned}
-\frac{B_y}{y}\bigg\{&\mathcal{Q}_x + \mathcal{Q}_B B_x + \mathcal{Q}_{B_x} B_{xx} + \mathcal{Q}_{B_y} B_{xy} + \mathcal{Q}_{B_\rho} B_{x\rho} + \mathcal{Q}_{B_{\rho\rho}} B_{x\rho\rho}\\
&+\frac{\mathcal{Q}_{B_{x\rho\rho}}}{yB_y}[(B_{xx}B_{y\rho\rho} - B_{xy}B_{y\rho\rho})y - B_x B_{x\rho\rho} - B_{xx}B_{\rho\rho}]\bigg\}\\
+\frac{B_x}{y}\bigg\{&\mathcal{Q}_y + \mathcal{Q}_B B_y + \mathcal{Q}_{B_x} B_{xy} + \mathcal{Q}_{B_y} B_{yy} + \mathcal{Q}_{B_\rho} B_{y\rho} + \mathcal{Q}_{B_{\rho\rho}} B_{y\rho\rho}\\
&+\frac{\mathcal{Q}_{B_{y\rho\rho}}}{yB_x}[(B_{yy}B_{x\rho\rho} - B_{xy}B_{y\rho\rho})y + B_{xy}B_{\rho\rho} + B_y B_{x\rho\rho}\\
&+2B_x B_{y\rho\rho} - 2B_y B_{x\rho\rho} - 2B_x B_{\rho\rho}/y]\bigg\}\\
-\kappa\bigg\{&\mathcal{Q}_\rho + \mathcal{Q}_B B_\rho \mathcal{Q}_{B_x} B_{x\rho} + \mathcal{Q}_{B_y} B_{y\rho} + \mathcal{Q}_{B_\rho} B_{\rho\rho}\\
&+\frac{\mathcal{Q}_{B_{\rho\rho}}}{\kappa y^2}[(B_x B_{y\rho\rho} - B_y B_{x\rho\rho})y - B_x B_{\rho\rho}]\bigg\}.
\end{aligned} \tag{2.25}$$

Now, since we know that $\mathcal{Q}$ depends on at most only one second-order derivative of B, $B_{\rho\rho}$, we know that the coefficients of all other second-order derivatives of B must be zero. The coefficients of B_{xx}, B_{xy}, B_{yy}, $B_{x\rho}$, $B_{y\rho}$ are, respectively,

$$-\frac{B_y}{y}\left\{\mathcal{Q}_{B_x} + \frac{\mathcal{Q}}{B_y}\left(B_{y\rho\rho} - \frac{B_{\rho\rho}}{y}\right)\right\} = 0 \tag{2.26a}$$

$$-\frac{B_y}{y}\left\{\mathcal{Q}_{B_y} - \mathcal{Q}_{B_{x\rho\rho}}\frac{B_{x\rho\rho}}{B_y}\right\} + \frac{B_x}{y}\left\{\mathcal{Q}_{B_x} + \frac{\mathcal{Q}_{B_{y\rho\rho}}}{B_x}\left(\frac{B_{\rho\rho}}{y} - B_{y\rho\rho}\right)\right\} = 0 \tag{2.26b}$$

$$\frac{B_x}{y}\left\{\mathcal{Q}_{B_y} + \mathcal{Q}_{B_{y\rho\rho}}\frac{B_{x\rho\rho}}{B_x}\right\} = 0 \tag{2.26c}$$

$$\frac{B_y}{y}\mathcal{Q}_{B_\rho} + \kappa\mathcal{Q}_{B_x} = 0 \tag{2.26d}$$

$$\frac{B_x}{y}\mathcal{Q}_{B_\rho} - \kappa\mathcal{Q}_{B_y} = 0. \tag{2.26e}$$

The solutions of (2.26*d*) and (2.26*e*) are easy to find and are inconsistent. We necessarily conclude that

$$\mathcal{Q}_{B_x} = \mathcal{Q}_{B_y} = \mathcal{Q}_{B_\rho} = 0 \tag{2.27}$$

and then from (2.26*a*)–(2.26*c*) we find

$$\mathcal{Q}_{B_{x\rho\rho}} = \mathcal{Q}_{B_{y\rho\rho}} = 0. \tag{2.28}$$

Collecting results we have

$$-\frac{B_y}{y}\mathcal{Q}_x + \frac{B_x}{y}\mathcal{Q}_y - \kappa\left\{\mathcal{Q}_\rho + \mathcal{Q}_B B_\rho - \mathcal{Q}_{B_{\rho\rho}}\frac{B_x B_{\rho\rho}}{\kappa y^2}\right\} = 0. \tag{2.29}$$

Since we know that $\mathcal{Q}$ does not depend on any first-order derivative of B we know that the coefficients of each must be zero. Hence, from the coefficients of B_x, B_y and B_ρ, respectively,

$$\mathcal{Q}_y/y + B_{\rho\rho}\mathcal{Q}_{B_{\rho\rho}}/y^2 = 0 \tag{2.30a}$$

$$-\mathcal{Q}_x/y = 0 \tag{2.30b}$$

$$-\kappa\mathcal{Q}_B = 0 \tag{2.30c}$$

leaving

$$\mathcal{Q}_\rho = 0. \tag{2.31}$$

From (2.30*b*) and (2.30*c*) we conclude that

$$\mathcal{Q}_x = \mathcal{Q}_B = 0. \tag{2.31}$$

In summary: we have shown that $\mathcal{Q}$ can depend on only y and $B_{\rho\rho}$. The general solution of (2.30*a*) is

$$\mathcal{Q} = Q(y/B_{\rho\rho}) \tag{2.32}$$

i.e., the only MCQ of third order or less admitted by (2.16) is potential vorticity (cf (1.11)). We reach a similar conclusion to that at the end of section 2.1: if (exact) MCQs do exist for our diffusive thermocline system they are probably not helpful as a solution ansatz.

3. Approximate MCQs

The analysis of the previous sections shows that whilst in some cases one can have success in determining materially conserved quantities of, for example, diffusive systems, which one can fruitfully use to construct exact solutions, it seems likely that more often than not one is confronted with a perturbation for which one is not so fortunate. In this section we show how one can 'make fortune' by considering quantitites which are nearly conserved. Specifically we answer:

(1) Can one determine solutions of systems in which non-ideal effects are not negligible (but small) from the solutions (obtained by means of MCQs) of systems in which they are?

As a consequence of the method used to answer this first question we are also able to indicate how one may answer the related question:

(2) Sometimes one can integrate the relation obtained between the MCQs for any function (e.g., that in example 3.1.). In other cases this is not the case (e.g., equation (1.12)) and one would therefore like to be able to determine the function without integration of the relation, by some systematic means. How can one do this?

Let us return to the thermocline system for illustration. In a real system modelled by the thermocline equations diffusion, friction and other effects are present, and these may often be represented by adding a term to the rhs of (1.10*b*), namely,

$$\frac{1}{y}(B_x B_{y\rho\rho} - B_y B_{x\rho\rho}) - \frac{1}{y^2}B_x B_{\rho\rho} = \mathcal{D} \tag{3.1}$$

where $\mathcal{D}$ is a differential function of B and is small except perhaps in boundary layers or frontal regions. With non-ideal effects added the equation relating q, B and ρ, (1.12), no longer satisfies—nor is implied by—the governing equations, or equivalently (3.1); it is only approximately true. We generalize by supposing

$$q = \frac{y}{B_{\rho\rho}} = \mathcal{Q}(B, \rho, X, Y) \qquad X = \delta_1 x \quad Y = \delta_2 y \quad |\delta_1|, |\delta_2| \ll 1 \tag{3.2}$$

where we have introduced the 'slow' variables X and Y: we are assuming that the functional relationship between the materially conserved quantities changes slowly with latitude and longitude. Of course (3.2) is not in general consistent with (3.1)—if (3.2) is used to eliminate the third-order terms in (3.1), whilst terms of O(1) identically cancel, terms of $O(\delta_1)$ and $O(\delta_2)$ remain—these new terms balance the 'new' terms on the right, i.e., those represented by $\mathcal{D}$, and hence we obtain a differential constraint on $\mathcal{Q}$ the solution of which yields some information about what the function $\mathcal{Q}$ might be.

Before proceeding further we must justify (3.2). In the ideal case we know that q is related to B and ρ by (1.12) and that, within a body of water from a particular source, $\mathcal{Q}$ is the same everywhere. If we introduce diffusion into the system then (1.12) is no longer correct, however, within a sufficiently small neighbourhood of the solution domain it is a good approximation. Furthermore, it is a good approximation in all other sub-domains, albeit with a different $\mathcal{Q}$ and the difference depends upon (the size of) κ. Taking the limit in which these (tesselated) sub-domains become small, this is precisely saying that

$$q = \mathcal{Q}(B, \rho, \alpha(x, \kappa), \beta(y, \kappa)). \tag{3.3}$$

Now, α and β are surely smooth functions of their respective arguments, i.e., *in coordinate-space adding diffusion is a singular perturbation, but in 'MCQ-space' the perturbation appears regular*. Hence both $\alpha(0, 0)$ and $\beta(0, 0)$ are constant and finite, so that—taking these constants to be zero as we may do, without loss of generality—we may write

$$\begin{aligned} \alpha(x, \kappa) &= \alpha_{11}\kappa x + \alpha_{21}\kappa^2 x + \alpha_{12}\kappa x^2 + \cdots \\ \beta(y, \kappa) &= \beta_{11}\kappa y + \beta_{21}\kappa^2 y + \beta_{12}\kappa y^2 + \cdots. \end{aligned} \tag{3.4}$$

If κ is small, as it is in the ocean interior, then we expect to neglect $O(\kappa^2)$ and higher-order terms. It is not obvious how to justify neglecting the remaining terms of degree two and higher in both x and y—at least in the general case. However, if the terms generated on the lhs by the 'slow' variables balance the new terms on the rhs, i.e., they are of the same order, then surely we are justified in this neglect. (Certainly we should not expect (3.2) to be the correct ansatz for all $\mathcal{D}$. We continue this discussion in section 4.)

We now show that ansätze of the form of (3.2) are successful if diffusion is added to both the 2D-Euler equations and to the thermocline system described above.

3.1. Approximately conserved quantities: the 2D-Euler equations

If viscosity is not negligible then the 2D-Euler equations, (1.3), are usually modified by the addition of diffusive terms giving (2.10). Combining these (and the zero z-component equation) into a single vector equation and taking the curl to obtain an equation for viscosity we find that only the z-component is non-zero:

$$\frac{D\omega}{Dt} = (\boldsymbol{u} \cdot \boldsymbol{\nabla})\omega = -\nu\nabla^2\omega \tag{3.5}$$

i.e., ω is no longer a MCQ, but is nearly materially conserved. The exact functional relationship between the streamfunction and velocity, (1.6), no longer satisfies, nor is implied by, the governing equations. Instead, following the ideas discussed above we suppose that

$$\omega = \mathcal{F}(\psi, X, Y) \qquad X = \nu x \qquad Y = \nu y. \tag{3.6}$$

Substituting (3.6) into (3.5) the O(1) terms cancel as expected and after using (1.4) the remaining terms are

$$\begin{aligned} \psi_x \mathcal{F}_Y - \psi_y \mathcal{F}_X &= -\nabla^2\omega \\ &= \psi_{xxxx} + 2\psi_{xxyy} + \psi_{yyyy} \\ &= \nabla^2\mathcal{F} + O(\nu). \end{aligned} \tag{3.7}$$

One now has the option of determining the flow by integrating (3.6) and (3.7) rather the governing equations, (2.10).

Example 3.1. *A classical problem which can be solved by our ideas is described by Batchelor (1956, section 3). Suppose there exists a 2D, steady, laminar flow within a fixed circular boundary at high Reynolds number. The inviscid equations, (1.3), are valid everywhere except in a boundary layer. How can one determine the interior flow?*

It is well known (e.g., Batchelor 1967) that the inviscid equations, (1.3), are not sufficient to determine the velocity field with no-normal-flow boundary conditions applied—the vorticity may vary arbitrarily from one streamline to another, i.e., $\mathcal{F}$ remains free. (Within closed streamlines imposing far-field conditions to fix $\mathcal{F}$ is not an option.) Batchelor makes use of integral constraints and finds that within a closed streamline, away from boundaries (i.e., in the region in which viscosity may be neglected), in the limit in which $\nu \to 0$ then $\omega = \omega_0$, constant, i.e., the fluid moves as a solid body. What happens if we make use of (3.7), and consider ν small, but not necessarily tending to zero?

Changing to plane-polar coordinates (r, θ) in which $\boldsymbol{u} = (u^r, u^\theta)$ and assuming rotational symmetry, i.e., $u^r = 0$, then (3.5) becomes

$$\left(\frac{u^\theta}{r}\frac{\partial}{\partial\theta}\right)\mathcal{F}(\psi) = \nu\frac{\partial}{\partial r}\nabla^2 u^\theta. \tag{3.8}$$

Following (3.6) we generalize $\mathcal{F}(\psi) \to \mathcal{F}(\psi, R, \Theta)$, where (R, Θ) are 'slow' coordinates so that $\partial/\partial\theta \to \partial/\partial\theta + \nu\partial/\partial\Theta$ and $\partial/\partial r \to \partial/\partial r + \nu\partial/\partial R$. Then using these in (3.8) the $\mathrm{O}(1)$ *terms yield $u^\theta r^{-1}\mathcal{F}_\psi\psi_\theta = 0$ which is satisfied since $\psi_\theta = u^r = 0$ and from the* $\mathrm{O}(\nu)$ *terms we find*

$$\frac{u^\theta}{r}\mathcal{F}_\Theta = \left(\frac{\partial^3}{\partial r^3} - \frac{1}{r^2}\frac{\partial}{\partial r} + \frac{1}{r}\frac{\partial^2}{\partial r^2}\right)u^\theta = -\nabla^2\omega. \tag{3.9}$$

Symmetry tells us to expect $\mathcal{F}_\Theta = 0$ hence integrating twice w.r.t. r we obtain the result

$$\omega = \omega_1(R)\ln(r) + \omega_0(R) \tag{3.10}$$

where both ω_1 and ω_0 are functions of integration. These may be found by matching the flow within the nearly inviscid region to fully viscous flow outside, for example, matching both ω and $\partial\omega/\partial r$—of course to prevent a singularity occurring at $r = 0$ we obviously require $\omega_1(0) = 0$. (In the limit in which $\nu \to 0$ then consistency with Batchelor's result indicates that $\omega_1 \to 0$ as $\nu \to 0$, i.e., ω_1 appears to provide a correction term in the case in which ν is small, but not tending to zero.)

Finally one can determine ψ. Since $\psi_\theta = 0$ then $\omega = -\partial^2\psi/\partial r^2 - (1/r)\partial\psi/\partial r$ (cf (1.5)). Hence integrating twice w.r.t. r we obtain

$$\psi(r) = c_0(R) + c_1(R)\ln(r) + \tfrac{1}{4}\omega_1(R)r^2[1 - \ln(r)] - \tfrac{1}{4}\omega_0(R)r^2 \tag{3.11}$$

where c_0 and c_1 are functions of integration, and $c_1(0) = 0$. Eliminating r between (3.10) and (3.11) gives $\mathcal{F}$.

3.2. Approximately conserved quantities: the thermocline equations

Let us continue our analysis of the diffusive thermocline system described by (2.16) with (1.10*a*). Following the ideas described above we suppose that the three quantities materially conserved in the ideal case, (1.11), are approximately functionally related, i.e., (3.2) is valid.

As a practical focus for our ideas we will consider a particular problem which arises in geophysical fluid dynamics. One might expect that the temperature in the world's oceans would decrease approximately linearly from the warm surface values to the abyssal values

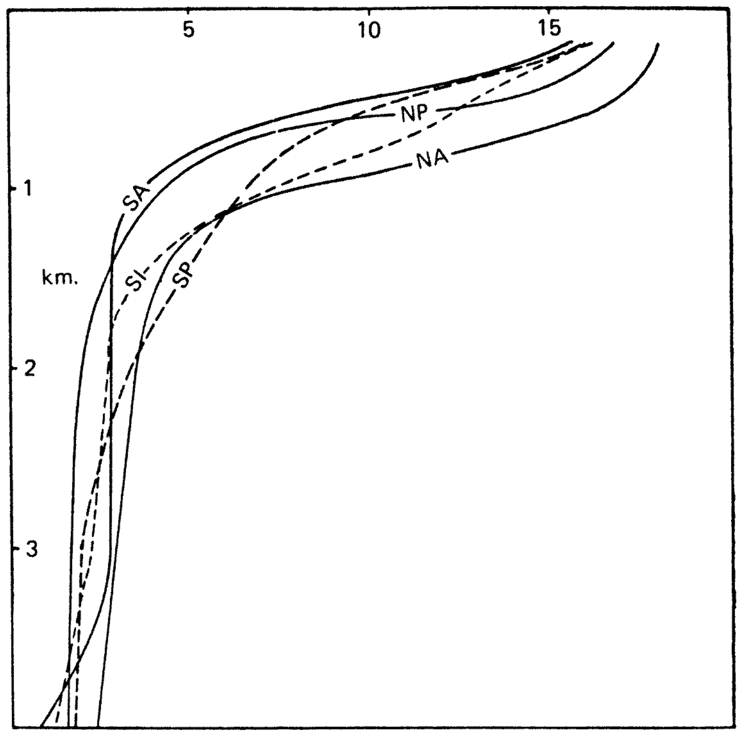

Figure 1. Characteristic temperature profiles in the worlds oceans: A = Atlantic, I = Indian, P = Pacific; N = north, S = south. Below the top few tens or hundreds of metres in which the temperature is roughly uniform (owing to turbulence) the temperature changes increasingly rapidly toward a maximum gradient at a mid depth of about 800–1000 m. Below this the gradient decreases. In the bottom 2 or 3 km the temperature is almost constant. (From Robinson and Stommel 1959.)

which are just above 0°C. In fact this is far from what happens—see figure 3.1. The question of what dynamics are responsible for this unexpected temperature profile has been addressed by many authors (see Hood and Williams (1996) for a survey of authors). Our approach is mathematically straightforward (given the ansatz) yet remarkably general—strong boundary conditions may be satisfied in contrast to many earlier analytical attempts at this so-called *thermocline problem*. (For a detailed description of the thermocline problem and a comprehensive survey of earlier publications see Hood and Williams (1996).)

We focus on the mathematics underlying the simplest model used to explain this: the thermocline equations are a hyperbolic system (Huang 1988), hence assuming just one source of fluid—so that that $\mathcal{Q}$ is the same throughout the solution domain—and prescribing inflow boundary conditions from the surface one obtains a well-posed problem. In short, the model suggests that the surface temperature field is advected downwards into the interior, deformed by the (horizontal) velocity field and also modified by diffusion: this sets up the unexpected temperature profile. This model is often called *ventilated* (e.g., Killworth 1987). (The significance of diffusion is the subject of considerable debate.) Note that aside from areas of ice formation or melting the greatest influence upon density in the ocean is temperature and to a good approximation (Bryan and Cox 1972) one may assume they are linearly related. Hence knowing the density field is equivalent to knowing the temperature field—of course temperature increases upwards whilst density increases downwards.

In a ventilation model addressing the thermocline problem it is usual to prescribe the vertical velocity at the 'surface' (in flow)

$$w_{\mathrm{E}}(x, y) = \frac{1}{y}(B_x B_{y\rho} - B_y B_{x\rho}) + \mathrm{O}(\kappa) \qquad \text{on} \quad z = B_\rho = 0 \tag{3.12}$$

and also the density field,

$$\rho = \rho_{\mathrm{s}} \qquad \text{on} \quad z = B_\rho = 0. \tag{3.13}$$

(These represent, implicitly, the true surface dynamics which are dominated by wind stress. w_{E} is negative (downwards) everywhere owing to the convergence of water at the surface acting under the influence of (average) wind stress—*Ekman pumping*.) This region is assumed to over-lie a region in which velocity is comparatively small—assumed negligible in the model. Therefore we set

$$B = 0 \qquad \text{on} \quad \rho = 0. \tag{3.14}$$

(Choosing the zero isopycnal to be the bottom is without loss of generality because the governing equations admit the point symmetry $\rho \to \rho + \rho_0$, ρ_0 an arbitrary constant.) The question of prescribed eastern conditions is discussed in the examples below. Finally, since the governing equations are hyperbolic the western, northern and sourthern boundaries are passive.

We consider two examples. (Full analysis of each example leading to (plotted) solutions of ocean velocity and density fields is beyond the scope of this paper. Rather, our aim is to show that (3.2) will satisfy the dynamics and boundary conditions, is flexible and practicable, and yields physical results, and in so doing answers the two questions posed at the beginning of this section.)

Example 3.2. *Generalized two-layer model ofthe ocean thermocline. Two-layer models are the simplest attempt to represent the thermocline in ocean models: here one assumes two layers in which the density (temperature) is uniform throughout, with a sharp change in density at the interface. Salmon (1994) generalized this idea by making the ansatz,*

$$q = \mathcal{Q}(\rho) \tag{3.15}$$

which is a special case of (1.12). (In fact there is good reason to believe that in some parts of the ocean, particularly away from lateral boundaries, this is a good approximation to the true state of the ocean, see Rhines and Young (1982).) Salmon prescribed $\mathcal{Q}$. In this example we address the question, what should $\mathcal{Q}$ be so that it is consistent with the dynamics (after diffusion is added)?

Following (1.4) we generalize by supposing that

$$q = \mathcal{Q}(\rho, X, Y). \tag{3.16}$$

In this example we may directly integrate our ansatz, (3.16), for any function $\mathcal{Q}$: we obtain

$$B(x, y, \rho, X, Y) = y\mathcal{F}(\rho, X, Y) + \rho b(x, y, X, Y) + a(x, y, X, Y) \tag{3.17}$$

where a and b are functions of integration, and

$$\mathcal{F}_{\rho\rho} = \frac{1}{\mathcal{Q}}. \tag{3.18}$$

Substituting (3.17) into (2.16) $\mathrm{O}(1)$ *terms cancel as expected, and from* $\mathrm{O}(\delta)$ *and* $\mathrm{O}(\kappa)$ *terms we find*

$$(\rho b_x + a_x)\delta_2 \mathcal{F}_{Y\rho\rho} - (\mathcal{F} + \rho b_y + a_y)\delta_1 \mathcal{F}_{X\rho\rho} = \kappa y \mathcal{F}_{\rho\rho\rho}. \tag{3.19}$$

(We have not neglected $\mathrm{O}(\kappa^2)$ *and similar terms—there are none.)*

We must interpret (3.19) carefully: a and b both depend upon x and y; $\mathcal{F}$ does not. We find that necessarily $\mathcal{F}_{X\rho\rho} = 0$, so that

$$\mathcal{F} = F(\rho, Y) + \rho f_1(X, Y) + f_0(X, Y) \tag{3.20}$$

where F, f_0 and f_1 are to be determined. Then comparison of (3.20) with (3.17) shows that we may take both $f_0 = f_1 = 0$ without loss of generality (by redefining a and b), i.e., we may assume that

$$\mathcal{F}_X = 0 \tag{3.21}$$

hence

$$(\rho b_x + a_x)\delta_2 F_{Y\rho\rho} = \kappa y F_{\rho\rho\rho}. \tag{3.22}$$

Now, assuming $F_Y \neq 0$ we find

$$\begin{aligned} a(x, y, Y) &= xya_{11}(Y) + a_0(y, Y) \\ b(x, y, Y) &= xyb_{11}(Y) + b_0(y, Y). \end{aligned} \tag{3.23}$$

Given this constraint on a and b is not clear that the prescribed surface boundary conditions can be satisfied, i.e., is our assumption, (3.16), incompatible with these? In fact one can establish the existence of a, b and F satisfying our (surface) boundary conditions given some reasonable assumptions: suppose that

$$\begin{aligned} \rho_{\mathrm{s}} &= r_{00} + r_{10}x + r_{01}y + r_{20}x^2 + \cdots \\ w_{\mathrm{E}} &= e_{00} + e_{10}x + e_{01}y + x_{20}x^2 + \cdots \end{aligned} \tag{3.24}$$

and that

$$F_{\rho\rho} = \Phi(\xi) = \sum_{n=0}^{\infty} \phi_n \xi^n \qquad \phi_n \to 0 \qquad \text{as} \quad n \to 0 \tag{3.25}$$

which are surely reasonable for physical solutions. Then expanding b_0 in powers of y, equating coefficients of products of powers of both x and y in both (3.13) and (3.12), one obtains a linear algebraic system of equations for ϕ_n and the coefficients in the expansion of b_0. It is easy to see from this that by keeping sufficient terms in the expansion of Φ one can both satisfy (3.13) and (3.12) to arbitrary degree in both x and y, and (since $\phi_n \to 0$ as $n \to 0$) that remaining terms in both (3.13) and (3.12) can be made arbitrarily small. (Note, given (3.15) one cannot prescribe both w_{E} and ρ_{s} arbitrarily. First, substituting (3.17) into (3.12) we find $w_{\mathrm{E}} = b_{11}y^{-1}(\rho F_\rho - F + F(0))$ so that b_{11} is non-zero, i.e., $b_x \neq 0$. Then, substituting (3.17) into (3.13) yields $yF_\rho + b = 0$ with $\rho = \rho_{\mathrm{s}}(x, y)$, so that $\rho_{\mathrm{s},x} \neq 0$. Secondly, (3.12) shows that in fact ρ_{s} and w_{E} are algebraically related—though this relation is not fixed.)

Given that solutions exist for prescribed boundary conditions one can will usually proceed from (3.22) and (3.23) numerically, i.e., determine F from the boundary conditions which include the 'surface' density field. In the context of Salmon's model, however, one one does not prescribe the surface density, instead one prescribes F. First we integrate (3.22). Using (3.14) we find that $a_x = 0$ and then the general solution is

$$F_{\rho\rho} = \Phi(\xi) \qquad \xi = \rho^2 + 2k \int \frac{\mathrm{dY}}{b_{11}(Y)} \qquad k = \kappa/\delta_2. \tag{3.26}$$

Since the perturbation in Y is regular then $b_{11} = b_{110} + b_{111}Y + \mathrm{O}(Y^2)$, hence

$$\xi = \rho^2 + \frac{2kY}{b_{110}} + \mathrm{O}(Y^2) \tag{3.27}$$

(we have set the constant of integration to zero w.l.o.g.).

It is worth considering a simple, particular case to gain some physical intuition about these solutions. Suppose

$$\Phi = \phi_0 + \phi_1 \xi \tag{3.28}$$

ϕ_0 *and* ϕ_1 *being constants which are easily determined: in the limit in which* κ, *and therefore* Y *vanish we expect a front to appear (cf), i.e.,* $F_{\rho\rho} = z_\rho = 0$ *on some isopycnal, and from (3.26) with (3.28) this isopycnal is given by* $\rho = \sqrt{-\phi_0/\phi_1}$; *then* ϕ_1 *gives the change in density between the upper and lower regions—so* ϕ_0 *and* ϕ_1 *are the model parameters giving position and size of jump in density between layers in the generalized two-layer model.*

(In this example the subject of the eastern boundary condition was deliberately avoided: the solution ansatz used cannot, in the context of the governing equations (2.16) with (1.10a), satisfy a strong eastern boundary condition such as no flow through $x = 0$ *(choosing* $x = 0$ *as the eastern boundary w.l.o.g.), only weak, integral conditions can be satisfied. In his generalized two-layer model Salmon (1994) uses modified governing equations.)*

Example 3.3. *A different approach to modelling the thermocline has been taken by Killworth (1987). Killworth supposed that*

$$B_{\rho\rho} = y\mathcal{F}(\rho)B \tag{3.29}$$

a second special case of (1.12). This ansatz was chosen specifically to satisfy no-flow through the eastern boundary at all latitudes and depths—one simply prescribes

$$B = 0 \qquad \textit{on} \quad x = 0. \tag{3.30}$$

In order to make analytical progress Killworth supposed that

$$\rho_{\mathrm{s},x} = 0. \tag{3.31}$$

From (3.13) we have $B_\rho(x, y, \rho_\mathrm{s}(x, y)) = 0$ *and computing the first-order partial derivatives of this w.r.t.* x *and* y *and using (3.31) we find* $B_{y\rho} + \rho_{\mathrm{s},y} B_{\rho\rho} = 0$ *and* $B_{x\rho} = 0$. *Hence (3.12) becomes*

$$\begin{aligned} w_\mathrm{E}(x, y) &= -\frac{1}{y}\rho_{\mathrm{s},y} B_x B_{\rho\rho} \\ &= -\rho_{\mathrm{s},y} B_x \mathcal{F} B \qquad \textit{on } \rho = \rho_s(y). \end{aligned} \tag{3.32}$$

Integrating w.r.t. x *one obtains an equation linking* B, $\mathcal{F}$ *and the prescribed surface boundary conditions. Solution yields* B *from which the velocity and depth fields can be found, by use of (1.10a).*

Can one make the same kind of analytical progress for the diffusive thermocline system, with an ansatz similar to (3.29)?

Following (1.14), and in the light of (3.21), we suppose

$$B_{\rho\rho} = y\mathcal{F}(\rho, Y)B. \tag{3.33}$$

We can use (3.33) to eliminate second-order ρ *derivatives from (2.16): computing each of the three first-order partial derivatives of (3.33) and eliminating the third-order derivatives of (2.16) the* $\mathrm{O}(1)$ *terms vanish as expected and equating the* $\mathrm{O}(\delta)$ *and* $\mathrm{O}(\kappa)$ *terms we find*

$$B_x \mathcal{F}_Y = ky(\mathcal{F}B_\rho + \mathcal{F}_\rho B) \qquad k = \kappa/\delta. \tag{3.34}$$

It remains to solve (3.33) simultaneously with (3.34). First, it turns out to be helpful to check consistency of these two equations by means of a horizontal expansion: since our governing equations contain only 'vertical' (no horizontal) diffusion we expect that solutions will be smooth in both x *and* y *so we suppose*

$$B(x, y, \rho, Y) = \sum_{i=0}^{I} \sum_{j=0}^{J} B_{ij}(\rho, Y) x^i y^j \tag{3.35}$$

where $B_{ij}(\rho, Y)$ are to be determined. Substituting (3.35) into both (3.33) and (3.34) leads to two recurrence relations for B_{ij}. It is not difficult to show by using these that physical solutions exist only if

$$\mathcal{F} = \frac{1}{\rho}\Phi(\xi) \qquad \xi = \frac{1}{2}\rho^2 + k\int \frac{\mathrm{d}Y}{b_{11}(Y)} = \frac{1}{2}\rho^2 + k\hat{b}_{11}Y + \mathrm{O}(Y^2) \qquad (3.36)$$

where $\rho b_{11}(Y) = B_{11}(\rho, Y)$ and $\hat{b}_{11}$ is constant (cf 3.26). Finally we can eliminate B_x between (3.34) and (3.32), and use (3.36) for $\mathcal{F}$. We find

$$\{(\rho^2\Phi_\xi - \Phi)B + \rho\Phi B_\rho\}y\rho_{\mathrm{s},y}\Phi B + \hat{b}_{11}\rho^2 w_{\mathrm{E}}\Phi_\xi = 0 \qquad \text{on } \rho = \rho_s(y). \qquad (3.37)$$

B and Φ (and therefore $\mathcal{F}$) are now found by simultaneous solution of (3.33) and (3.37), a pair of ordinary differential equations, which are subject to the bottom condition, (3.14), and the surface conditions (3.12) and (3.13). (In fact one needs an additional 'vertical' condition—for example one could specify the behaviour of $\mathcal{F}$ as $\rho \to 0$.) Again the extra information gained by making our ansatz based on MCQs of the ideal system is enough to determine the unknown function. (In Killworth's analysis solutions existed only for $w_{\mathrm{E}} < 0$. In our case solutions apparently exist the case $w_{\mathrm{E}} > 0$ but it is not obvious that these will be physical since with $w_{\mathrm{E}} > 0$ the density field is prescribed at an out-flow boundary of the solution domain.)

4. Discussion

Let us first return to the questions posed at the beginning of section 3. First, we have shown it *is* possible to successfully construct solutions of non-ideal systems by supposing that a singular perturbation in coordinate space is regular in MCQ-space: given a solution-ansatz based on a functional relationship between exact MCQs of an ideal system one supposes that a similar relationship holds for the non-ideal system, the difference being that the function is not the same throughout the solution domain, rather it 'slowly' changes with each coordinate. The solutions satisfy full boundary conditions. Secondly, in three cases we have shown that it is possible to determine the function relating MCQs of an ideal system by supposing that such a system is a limiting case of a non-ideal system. Whilst integral methods have been used to address the same problem (Batchelor 1956, Rhines and Young 1982) these require closed streamlines, the method introduced here does not.

When might the methods break down? First, if there are two (or more) sources of fluid. Consider the thermocline problem described in section 3.2. We focused on ventilated models, i.e., those in which water enters the solution domain from above. Alternative models suppose that in addition there is upwelling from below and that water from the two sources meets 'in the middle'. To address this situation one would need to solve a ventilated problem in the upper part of the solution domain, a similar (upsidedown!) problem in the lower part and match at the interface. Care is needed *at* the interface: here ρ_z is at a maximum and therefore z_ρ is at a minimum so that $z_{\rho\rho} = B_{\rho\rho\rho} = 0$, i.e., the rhs (non-ideal terms) of (2.16) vanishes—our ansatz is therefore valid. However, if one adds Fickian diffusion to (1.8*b*), i.e., adds a rhs of $y^{-2}\kappa M_{zzzz}$ (note this is a different parametrization to that given by (2.16)) then our ansatz is probably not valid at the interface as this non-ideal term takes its *maximum* value there. (One might consider an inner problem in which horizontal motion is neglected, (see Salmon (1990), Hood and Williams (1996).)

Secondly, it is by no means obvious that our ansatz, or even a generalization of it, is valid for all non-ideal effects. First, recall (3.4): we were by no means rigorous in choosing $\alpha = \kappa x$ and $\beta = \kappa y$. Indeed the author's current work suggests that supposing, for example, $q = \mathcal{F}(\rho, Y, \hat{Y})$, $Y = \kappa y$, $\hat{Y} = \kappa^2 y$, leads to a useful second-order correction term in some problems (though such second-order terms had no place in the examples given above: in each, O(1) terms cancelled and O(κ) terms provided a constraint—there were no O(κ^2) terms). Further, initially sticking with the general form for β (and α) and letting the boundary conditions play a role in setting the particular form may prove useful. However, for some rhs a balance may be impossible to obtain. For illustration choose the rhs of (2.16) to be replaced with $\kappa y^2 B_{y\rho\rho\rho}$. The rhs now generates O(κ^2) terms which are not balanced on the lhs. Even supposing $\mathcal{F}(\rho, R, Y)$, $Y = \beta(\kappa, \rho)$, $R = \eta(\kappa, \rho)$ one cannot obtain a balance. It seems that this particular perturbation is singular, even in MCQ-space.

A next step is to continue with examples 3.2 and 3.3, applying realistic boundary conditions. The author is currently pursuing these ideas, with a further generalised ansatz which eliminates some of the restrictions found such as (3.23).

Here we have focused on the perturbation of steady, ideal systems by the addition of diffusion and other non-ideal effects. Can one use methods similar to those introduced above if the systems become unsteady, a particular case of adding an independent variable to the system? (In the case of the ideal thermocline equations the Bernoulli functional, B, is no longer materially conserved.) The author's current work indicates that methods closely related to those introduced above *will* be effective.

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